7.3: Translation and Partial Fractions

Let us refresh ourselves on the rules of **partial fraction decomposition.** For the next two rules let

$$R(s) = \frac{P(s)}{Q(s)}$$

where $P(s)$ is a polynomial of degree less than that of $Q(s)$.

Rule 1. (Linear Factor Partial Fractions)
The portions of the partial fraction decomposition of $R(s)$ corresponding to the linear factor $s - a$ of multiplicity n is a sum of n partial fractions, having the form

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}$$

where $A_1, A_2, \ldots, A_n$ are constants.

Rule 2. (Quadratic Factor Partial Fractions)
The portions of the partial fraction decomposition of $R(s)$ corresponding to the quadratic factor $(s-a)^2+b^2$ of multiplicity n is a sum of n partial fractions, having the form

$$\frac{A_1 s + B_1}{(s-a)^2 + b^2} + \frac{A_2 s + B_2}{((s-a)^2 + b^2)^2} + \cdots + \frac{A_n s + B_n}{((s-a)^2 + b^2)^n}$$

where $A_1, A_2, \ldots, A_n, B_1, B_2 \ldots, B_n$ are constants.

Theorem 1. (Translation of the s-Axis)
If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > c$, then $\mathcal{L}\{e^{at}f(t)\}$ exists for $s > a + c$, and

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a).$$

Equivalently,

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t).$$

Thus the translation $s \to s - a$ in the transform (s-axis) corresponds to multiplication of the original function of t by e^{at}.

Example 1. Find $\mathcal{L}\{e^{at}t^n\}$, $\mathcal{L}\{e^{at}\cos kt\}$ and $\mathcal{L}\{e^{at}\sin kt\}$.

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \rightarrow$ So $\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}$

$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2} \rightarrow$ So $\mathcal{L}\{e^{at}\cos kt\} = \frac{s-a}{(s-a)^2+k^2}$

$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} \rightarrow$ So $\mathcal{L}\{e^{at}\sin kt\} = \frac{k}{(s-a)^2+k^2}$

Example 2. Consider a mass-and-spring system with $m = \frac{1}{2}$, $k = 17$ and $c = 3$ which gives the equation

$$x'' + 6x' + 34x = 0.$$

If the mass is set into motion with $x(0) = 3$ and $x'(0) = 1$, find the resulting damped free oscillations.

$$\mathcal{L}\{x''+6x'+34x\} = \mathcal{L}\{0\}$$

$$0 = (s^2X - 3s - 1) + 6(sX - 3) + 34X$$

$$X = \frac{3s+19}{s^2+6s+34} = 3\cdot\frac{s+3}{(s+3)^2+25} + 2\cdot\frac{5}{(s+3)^2+25}$$

Thus

$$x = \mathcal{L}^{-1}\{X\}$$

$$= e^{-3t}(3\cos 5t + 2\sin 5t)$$

Example 3. Find the inverse Laplace transform of

$$R(s) = \frac{s^2+1}{s^3 - 2s^2 - 8s}.$$

$$\frac{s^2+1}{s^3-2s^2-8s} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-4} = \frac{-1/8}{s} + \frac{5/12}{s+2} + \frac{17/24}{s-4}.$$

So $\mathcal{L}^{-1}\{R(s)\} = -\frac{1}{8} + \frac{5}{12}e^{-2t} + \frac{17}{24}e^{4t}$.

Example 4. Solve the initial value problem

$$y'' + 4y' + 4y = t^2; \quad y(0) = y'(0) = 0.$$

$$\mathcal{L}\{y''+4y'+4y\} = \mathcal{L}\{t^2\}$$

$$\frac{2}{s^3} = s^2Y + 4sY + 4Y$$

$$Y = \frac{2}{s^3(s+2)^2} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{(s+2)^2} + \frac{E}{(s+2)}$$

$A = 1/2$, $B = -1/2$, $C = 3/8$, $D = -1/4$, $E = -\frac{3}{8}$

So that

$$y = \frac{1}{4}t^2 - \frac{1}{2}t + \frac{3}{8} - \frac{1}{4}te^{-2t} - \frac{3}{8}e^{-2t}$$

Exercise 1. Solve the damped and forced mass-spring-dashpot system given by

$$x'' + 6x' + 34x = 30\sin 2t; \quad x(0) = x'(0) = 0.$$

$$\mathcal{L}\{x''+6x'+34x\} = \mathcal{L}\{30\sin 2t\}$$

$$\frac{60}{s^2+4} = s^2X + 6sX + 34X$$

$$X = \frac{60}{(s^2+4)(s^2+6s+34)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s+3)^2+25}$$

$$= \frac{1}{29}\left(\frac{-10s+50}{s^2+4} + \frac{10s+10}{(s+3)^2+25}\right)$$

So

$$X = \frac{5}{29}(-2\cos 2t + 5\sin 2t) + \frac{2}{29}e^{-3t}(5\cos 5t - 2\sin 5t)$$

Exercise 2. Solve the initial value problem

$$y^{(4)} + 2y'' + y = 4te^t; \quad y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0.$$

$$\mathcal{L}\{y^{(4)}+2y''+y\} = \mathcal{L}\{4te^t\}$$

$$\frac{4}{(s-1)^2} = (s^4+2s^2+1)Y$$

$$Y = \frac{4}{(s-1)^2(s^2+1)^2} = \frac{A}{(s-1)^2} + \frac{B}{(s-1)} + \frac{Cs+D}{(s^2+1)^2} + \frac{Es+F}{s^2+1}$$

$$\left(\begin{array}{ll} A=1 & E=2 \\ B=-2 & F=1 \\ C=2 & \\ D=0 & \end{array}\right)$$

$$= \frac{1}{(s-1)^2} - \frac{2}{s-1} + \frac{2s}{(s^2+1)^2} + \frac{2s+1}{s^2+1}$$

So $y = (t-2)e^t + (t+1)\sin t + 2\cos t.$

Example 5. Solve the initial value problem

$$x'' + \omega_0^2 x = F_0 \sin \omega t; \quad x(0) = x'(0) = 0.$$

$$\mathcal{L}\{x'' + \omega_0^2 x\} = \mathcal{L}\{F_0 \sin \omega t\}$$

$$\frac{F_0 \omega}{s^2+\omega^2} = s^2 X + \omega_0^2 X$$

$$X = \frac{F_0 \omega}{(s^2+\omega^2)(s^2+\omega_0^2)}.$$

If $\omega \neq \omega_0$,

$$X = \frac{F_0 \omega}{\omega^2-\omega_0^2}\left(\frac{1}{s^2+\omega_0^2} - \frac{1}{s^2+\omega^2}\right)$$

and

$$x = \frac{F_0 \omega}{\omega^2-\omega_0^2}\left(\frac{1}{\omega_0}\sin\omega_0 t - \frac{1}{\omega}\sin\omega t\right)$$

If $\omega = \omega_0$,

$$X = \frac{F_0 \omega_0}{(s^2+\omega_0^2)^2}$$

and

$$x = \frac{F_0}{2\omega_0^2}\left(\sin\omega_0 t - \omega_0 t\cos\omega_0 t\right)$$

(Resonance)

Homework. 1-21, 27-35 (odd)